EFFECTIVE LAGRANGIANS FOR QCD,

DUALITY AND EXACT RESULTS

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Abstract   I briefly discuss effective Lagrangians for strong interactions while concentrating on two specific lagrangians for QCD at large matter density. I then introduce spectral duality in QCD a la Montonen and Olive. The latter is already present in QCD in the hadronic phase. However it becomes transparent at large chemical potential. Finally I show the relevance of having exact non perturbative constraints such as t'Hooft anomaly conditions at zero and nonzero quark chemical potential on the possible phases of strongly interacting matter. An important outcome is that for three massless quarks at any chemical potential the only non trivial solution of the constraints is chiral symmetry breaking. This also shows that for three massless flavors at large quark chemical potential CFL is the ground state.

1.   Effective Lagrangians for QCD

In the non perturbative regime of strongly interacting theories effective Lagrangians play a dominant role since they efficiently describe the non perturbative dynamics in terms of the relevant degrees of freedom. Symmetries, anomalous and exact, are used to constrain the effective Lagrangians. An important point is that the effective Lagrangian approach is applicable to any region of the QCD or QCD-like phase diagram whenever the relevant degrees of freedom and the associated symmetries are defined.

1.1   Zero temperature and quark chemical potential

At zero temperature and quark chemical potential the simplest effective Lagrangian describing a relevant part of the nonperturbative physics of the Yang-Mills (YM) theory is the glueball Lagrangian whose potential is [1–4]:

$$V = \frac{H}{2} \ln \left[\frac{H}{\Lambda^4} \right] \ .$$

(1)

The latter is constrained using trace anomaly and $H \sim \text{Tr}\left[G^{\mu\nu}G_{\mu\nu}\right]$ with $G^{\mu\nu}$ the gluon field stress tensor. It describes the vacuum of a generic Yang-Mills theory. A similar effective Lagrangian (using superconformal anomalies) can be written for the non perturbative super Yang-Mills (SYM) theory. This is the celebrated Veneziano-Yankielowicz [5] lagrangian. In Yang Mills and super Yang-Mills theories no exact continuous global symmetries are present which can break spontaneously and hence no goldstones are present. The situation is different when flavors are included in the theory. Here the spontaneous breaking of chiral symmetry leads to a large number of goldstone's excitations. We note that in [8, 9] we were able, using string techniques, to derive a number of fundamental perturbative and non perturbative properties for supersymmetric QCD such as the beta function, fermion condensate as well as chiral anomalies.

Recently in [6] we constructed effective Lagrangians of the Veneziano-Yankielowicz (VY) type for two non-supersymmetric but strongly interacting theories with a dirac fermion either in the two index symmetric or two index antisymmetrix representation of the gauge group. These theories are planar equivalent, at $N \rightarrow \infty$ to SYM [7]. In this limit the non-supersymmetric effective Lagrangians coincide with the bosonic part of the VY Lagrangian.

We departed from the supersymmetric limit in two ways. First, we considered finite values of N. Then $1/N$ effects break supersymmetry. We suggested the simplest modification of the VY Lagrangian which incorporates these $1/N$ effects, leading to a non-vanishing vacuum energy density. We analyzed the spectrum of the finite-N non-supersymmetric daughters. For $N = 3$ the two-index antisymmetric representation (one flavor) *is one-flavor QCD*. We showed that in this case the scalar quark-antiquark state is heavier than the corresponding pseudoscalar state, the η'. Second, we added a small fermion mass term which breaks supersymmetry explicitly. The vacuum degeneracy is lifted, the parity doublets split and we evaluated this splitting. The θ-angle dependence and its implications were also investigated. This new effective Lagrangian provides a number of fundamental results about QCD which can be already tested either experimentally or via lattice simulations.

This new type of expansion in the inverse of number colors in which the quark representation is the two index antisymmetric representation of the gauge group at any given N may very well be more convergent then the ordinary $1/N$ expansion. In the ordinary case one keeps the fermion in the fundamental representation of the gauge group while increasing the number of colors. Indeed recently in [10] we have studied the dependence on the number of colors (while keeping the fermions in the fundamental representation of the gauge group) of the leading pi pi scattering amplitude in chiral dynamics. We have demonstrated the existence of a critical number of colors for and above which the low energy pi pi scattering amplitude computed from the simple sum of the current algebra and vector meson terms is crossing symmetric and unitary at leading

order in a $1/N$ expansion. The critical number of colors turns out to be $N = 6$ and is insensitive to the explicit breaking of chiral symmetry. This means that the ordinary $1/N$ corrections for the real world are large.

1.2 Nonzero temperature and quark chemical potential

At nonzero temperature the center of the $SU(N)$ gauge group becomes a relevant symmetry [11]. However except for mathematically defined objects such as Polyakov loops the physical states of the theory are neutral under the center group symmetry.

A new class of effective Lagrangians have been constructed to show how the information about the center group symmetry is efficiently transferred to the actual physical states of the theory [12–15] and will be reviewed in detail elsewhere. Via these Lagrangians we were also able to have a deeper understanding of the relation between chiral restoration and deconfinement [15] for quarks in the fundamental and in the adjoint representation of the gauge group.

I will focus here on the two basic effective Lagrangians developed for color superconductivity. More specifically the lagrangian for the color flavor locked phase (CFL) of QCD at high chemical potential and the 2 flavor color superconductive effective Lagrangian.

A color superconducting phase is a reasonable candidate for the state of strongly interacting matter for very large quark chemical potential [16–20]. Many properties of such a state have been investigated for two and three flavor QCD. In some cases these results rely heavily on perturbation theory, which is applicable for very large chemical potentials. Some initial applications to supernovae explosions and gamma ray bursts can be found in [21] and [22] respectively, see also [23]. The interested reader can find a discussion of the effects of color superconductivity on the mass-radius relationship of compact stars in [45]

2. Color Flavor Locked Phase

For $N_f = 3$ light flavors at very high chemical potential dynamical computations suggest that the preferred phase is a superconductive one and the following ansatz for a quark-quark type of condensate is energetically favored:

$$\epsilon^{\alpha\beta} < q_{L\alpha;a,i}q_{L\beta;b,j} > \sim k_1\delta_{ai}\delta_{bj} + k_2\delta_{aj}\delta_{bi} . \tag{2}$$

A similar expression holds for the right transforming fields. The Greek indices represent spin, a and b denote color while i and j indicate flavor. The condensate breaks the gauge group completely while locking the left/right transformations with color. The final global symmetry group is $SU_{c+L+R}(3)$, and the low energy spectrum consists of 9 Goldstone bosons.

3.   Duality made transparent in QCD

Here we seek insight regarding the relevant energy scales of various physical states of the color flavor locked phase (CFL), such as the vector mesons and the solitons [24]. Our results do not support the naive expectation that all massive states are of the order of the color superconductive gap, Δ. Our strategy is based on exploiting the significant information already contained in the low–energy effective theory for the massless states. We transfer this information to the massive states of the theory by making use of the fact that higher derivative operators in the low–energy effective theory for the lightest state can also be induced when integrating out heavy fields. For the vector mesons, this can be seen by considering a generic theory containing vector mesons and Goldstone bosons. After integrating out the vector mesons, the induced local effective Lagrangian terms for the Goldstone bosons must match the local contact terms from operator counting. We find that each derivative in the (CFL) chiral expansion is replaced by a vector field ρ_μ as follows

$$\partial \to \frac{\Delta}{F_\pi}\rho \ . \tag{3}$$

This relation allows us to deduce, among other things, that the energy scale for the vector mesons is

$$m_v \sim \Delta \ , \tag{4}$$

where m_v is the vector meson mass. Our result is in agreement with the findings in [25, 26]. We shall see that this also suggests that the KSRF relation holds in the CFL phase.

In the solitonic sector, the CFL chiral Lagrangian [27, 28] gives us the scaling behavior of the coefficient of the Skyrme term and thus shows that the mass of the soliton is of the order of

$$M_{\text{soliton}} \sim \frac{F_\pi^2}{\Delta} \ , \tag{5}$$

which is contrary to naive expectations. This is suggestive of a kind of duality between vector mesons and solitons in the same spirit as the duality advocated some years ago by Montonen and Olive for the $SU(2)$ Georgi-Glashow theory [29]. This duality becomes more apparent when considering the product

$$M_{\text{soliton}}m_v \sim F_\pi^2 \ , \tag{6}$$

which is independent of the scale, Δ. In the present case, if the vector meson self-coupling is $\widetilde{g}$, we find that the Skyrme coefficient, $e \sim \Delta/F_\pi$, can be identified with $\widetilde{g}$. Thus, the following relations hold:

$$M_{\text{soliton}} \propto \frac{F_\pi}{\widetilde{g}} \quad \text{and} \quad m_v \propto \widetilde{g}\,F_\pi \ . \tag{7}$$

In this notation the electric-magnetic (i.e. vector meson-soliton) duality is transparent. Since the topological Wess-Zumino term in the CFL phase is identical to that in vacuum, we identify the soliton with a physical state having the quantum numbers of the nucleon. We expect that the product of the nucleon and vector meson masses will scale like F_π^2 for any non-zero chemical potential for three flavors. Interestingly, quark-hadron continuity can be related to duality [30]. Testing this relation can also be understood as a quantitative check of quark-hadron continuity. It is important to note that our results are tree level results and that the resulting duality relation can be affected by quantum corrections. Our results have direct phenomenological consequences for the physics of compact stars with a CFL phase. While vector mesons are expected to play a relevant role, solitons can safely be neglected for large values of the quark chemical potential.

3.1 The Lagrangian for CFL Goldstones

When diquarks condense for the three flavor case, we have the following symmetry breaking:

$$[SU_c(3)] \times SU_L(3) \times SU_R(3) \times U_B(1) \rightarrow SU_{c+L+R}(3) \ .$$

The gauge group undergoes a dynamical Higgs mechanism, and nine Goldstone bosons emerge. Neglecting the Goldstone mode associated with the baryon number and quark masses (which will not be important for our discussion at lowest order), the derivative expansion of the effective Lagrangian describing the octect of Goldstone bosons is [27, 28]:

$$\mathcal{L} = \frac{F_\pi^2}{8} \text{Tr} \left[\partial_\mu U \partial^\mu U^\dagger \right] \equiv \frac{F_\pi^2}{2} \text{Tr} \left[p_\mu p^\mu \right] \ , \tag{8}$$

with $p_\mu = \frac{i}{2} \left(\xi \partial_\mu \xi^\dagger - \xi^\dagger \partial_\mu \xi \right)$, $U = \xi^2$, $\xi = e^{i\frac{\phi}{F_\pi}}$ and ϕ is the octet of Goldstone bosons. U transforms linearly according to $g_L U g_R^\dagger$ and $g_{L/R} \in SU_{L/R}(3)$ while ξ transforms non-linearly:

$$\xi \rightarrow g_L \, \xi \, K^\dagger \left(\phi, g_L, g_R \right) \equiv K \left(\phi, g_L, g_R \right) \xi \, g_R^\dagger \ . \tag{9}$$

This constraint implicitly defines the matrix, $K \left(\phi, g_L, g_R \right)$. Here, we wish to examine the CFL spectrum of massive states using the technique of integrating in/out at the level of the effective Lagrangian. F_π is the Goldstone boson decay constant. It is a non-perturbative quantity whose value is determined experimentally or by non-perturbative techniques (e.g. lattice computation). For very large quark chemical potential, F_π can be estimated perturbatively. It is found to be proportional to the Fermi momentum, $p_F \sim \mu$, with μ the quark chemical potential [31]. Since a frame must be fixed in order to introduce a chemical

potential, spatial and temporal components of the effective Lagrangians split. This point, however, is not relevant for the validity of our results.

When going beyond the lowest-order term in derivatives, we need a counting scheme. For theories with only one relevant scale (such as QCD at zero chemical potential), each derivative is suppressed by a factor of F_π. This is not the case for theories with multiple scales. In the CFL phase, we have both F_π and the gap, Δ, and the general form of the chiral expansion is [31]:

$$L \sim F_\pi^2 \Delta^2 \left(\frac{\vec{\partial}}{\Delta} \right)^k \left(\frac{\partial_0}{\Delta} \right)^l U^m U^{\dagger n} \ . \tag{10}$$

Following [31], we distinguish between temporal and spatial derivatives. Chiral loops are suppressed by powers of $p/4\pi F_\pi$, and higher-order contact terms are suppressed by p/Δ where p is the momentum. Thus, chiral loops are parametrically small compared to contact terms when the chemical potential is large.

There is also a topological term which is essential in order to satisfy the t'Hooft anomaly conditions [32–34] at the effective Lagrangian level. It is important to note that respecting the t'Hooft anomaly conditions is more than an academic exercise. In fact, it requires that the form of the Wess-Zumino term is the same in vacuum and at non-zero chemical potential. Its real importance lies in the fact that it forbids a number of otherwise allowed phases which cannot be ruled out given our rudimentary treatment of the non-perturbative physics. As an example, consider a phase with massless protons and neutrons in three-color QCD with three flavors. In this case chiral symmetry does not break. This is a reasonable realization of QCD for any chemical potential. However, it does not satisfy the t'Hooft anomaly conditions and hence cannot be considered. Were it not for the t'Hooft anomaly conditions, such a phase could compete with the CFL phase.

Gauging the Wess-Zumino term with to respect the electromagnetic interactions yields the familiar $\pi^0 \to 2\gamma$ anomalous decay. This term [35] can be written compactly using the language of differential forms. It is useful to introduce the algebra-valued Maurer-Cartan one form $\alpha = \alpha_\mu dx^\mu = (\partial_\mu U) U^{-1} dx^\mu \equiv (dU) U^{-1}$ which transforms only under the left $SU_L(3)$ flavor group. The Wess-Zumino effective action is

$$\Gamma_{WZ}[U] = C \int_{M^5} \text{Tr} \left[\alpha^5 \right] \ . \tag{11}$$

The price which must be paid in order to make the action local is that the spatial dimension must be augmented by one. Hence, the integral must be performed over a five-dimensional manifold whose boundary (M^4) is ordinary Minkowski space. In [27, 32, 36] the constant C has been shown to be the

same as that at zero density, i.e.

$$C = -i\frac{N_c}{240\pi^2} \; , \tag{12}$$

where N_c is the number of colors (three in this case). Due to the topological nature of the Wess-Zumino term its coefficient is a pure number.

3.2   The vector mesons

It is well known that massive states are relevant for low energy dynamics. Consider, for example, the role played by vector mesons in pion-pion scattering [10, 37] in saturating the unitarity bounds. More specifically, vector mesons play a relevant role when describing the low energy phenomenology of QCD and may also play a role also in the dynamics of compact stars with a CFL core [38]. In order to investigate the effects of such states, we need to know their in–medium properties including their gaps and the strength of their couplings to the CFL Goldstone bosons. Except for the extra spontaneously broken $U(1)_B$ symmetry, the symmetry properties of the CFL phase have much in common with those of zero density phase of QCD. This fact allows us to make some non perturbative but reasonable estimates of vector mesons properties in medium. We have already presented the general form of the chiral expansion in the CFL phase. As will soon become clear, we are now interested in the four–derivative (non–topological) terms whose coefficients are proportional to

$$\frac{F_\pi^2}{\Delta^2} \; . \tag{13}$$

This must be contrasted with the situation at zero chemical potential, where the coefficient of the four–derivative term is always a pure number before quantum corrections are taken into account. In vacuum, the tree-level Lagrangian which simultaneously describes vector mesons, Goldstone bosons, and their interactions is:

$$
\begin{aligned}
L & = & \frac{F_\pi^2}{2}\text{Tr}\left[p_\mu p^\mu\right] + \frac{m_v^2}{2}\text{Tr}\left[\left(\rho_\mu + \frac{v_\mu}{\widetilde{g}}\right)^2\right] \\
& & - \frac{1}{4}\text{Tr}\left[F_{\mu\nu}(\rho)F^{\mu\nu}(\rho)\right] \; ,
\end{aligned}
\tag{14}
$$

where $F_\pi \simeq 132$ Mev and v_μ is the one form $v_\mu = \frac{i}{2}\left(\xi\partial_\mu\xi^\dagger + \xi^\dagger\partial_\mu\xi\right)$ with $U = \xi^2$ and $F_{\mu\nu}(\rho) = \partial_\mu\rho_\nu - \partial_\nu\rho_\mu + i\widetilde{g}\left[\rho_\mu, \rho_\nu\right]$. At tree level this Lagrangian agrees with the hidden local symmetry results [39].

When the vector mesons are very heavy with respect to relevant momenta, they can be integrated out. This results in the field constraint:

$$\rho_\mu = -\frac{v_\mu}{\widetilde{g}} \; . \tag{15}$$

Substitution of this relation in the vector meson kinetic term (i.e., the replacement of $F_{\mu\nu}(\rho)$ by $F_{\mu\nu}(v)$) gives the following four derivative operator with two time derivatives and two space derivatives [40]:

$$\frac{1}{64\,\widetilde{g}^2}\mathrm{Tr}\left[[\alpha_\mu,\alpha_\nu]^2\right]\ . \tag{16}$$

The coefficient is proportional to $1/\widetilde{g}^2$. It is also relevant to note that since we are describing physical fields we have considered canonically normalized fields and kinetic terms. This Lagrangian can also be applied to the CFL case. In the vacuum, $\widetilde{g}$ is a number of order one independent of the scale at tree level. This is no longer the case in the CFL phase. Here, by comparing the coefficient of the four–derivative operator in eq. (16) obtained after having integrated out the vector meson with the coefficient of the same operator in the CFL chiral perturbation theory we determine the following scaling behavior of $\widetilde{g}$:

$$\widetilde{g}\propto\frac{\Delta}{F_\pi}\ . \tag{17}$$

By expanding the effective Lagrangian with the respect to the Goldstone boson fields, one sees that $\widetilde{g}$ is also connected to the vector meson coupling to two pions, $g_{\rho\pi\pi}$, through the relation

$$g_{\rho\pi\pi}=\frac{m_v^2}{\widetilde{g}F_\pi^2}\ . \tag{18}$$

In vacuum $g_{\rho\pi\pi}\simeq 8.56$ and $\widetilde{g}\simeq 3.96$ are quantities of order one. Since v_μ is essentially a single derivative, the scaling behavior of $\widetilde{g}$ allows us to conclude that each derivative term is equivalent to $\widetilde{g}\,\rho_\mu$ with respect to the chiral expansion. For example, dropping the dimensionless field U, the operator with two derivatives becomes a mass operator for the vector meson

$$F_\pi^2\partial_\mu^2\to F_\pi^2\,\widetilde{g}^2\,\rho_\mu^2\sim\Delta^2\rho_\mu^2\ . \tag{19}$$

This demonstrates that the vector meson mass gap is proportional to the color superconducting gap. This non-perturbative result is relevant for phenomenological applications. It is interesting to note that our simple counting argument agrees with the underlying QCD perturbative computations of Ref. [25] and also with recent results of Ref. [26]. In [41], at high chemical potential, vector meson dominance is discussed. However, our approach is more general since it does not rely on any underlying perturbation theory. It can be applied to theories with multiple scales for which the counting of the Goldstone modes is known. Since $m_v^2\sim\Delta^2$, we find that $g_{\rho\pi\pi}$ scales with $\widetilde{g}$ suggesting that the KSRF relation is a good approximation also in the CFL phase of QCD.

3.3 CFL-Solitons

The low energy effective theory supports solitonic excitations which can be identified with the baryonic sector of the theory at non-zero chemical potential. In order to obtain classically stable configurations, it is necessary to include at least a four–derivative term (containing two temporal derivatives) in addition to the usual two–derivative term. Such a term is the Skyrme term:

$$L^{\text{skyrme}} = \frac{1}{32\,e^2}\text{Tr}\left[[\alpha_\mu, \alpha_\nu]^2\right] \ .$$
(20)

Since this is a fourth–order term in derivatives not associated with the topological term we have:

$$e \sim \frac{\Delta}{F_\pi} \ .$$
(21)

This term is the same as that which emerges after integrating out the vector mesons (see eq. (16)), and one concludes that $e = \sqrt{2}\,\widetilde{g}$ [40]. The simplest complete action supporting solitonic excitations is:

$$\int d^4x \left[\frac{F_\pi^2}{2}\text{Tr}\,[p_\mu p^\mu] + L^{\text{skyrme}}\right] + \Gamma_{WZ} \ .$$
(22)

The Wess-Zumino term in eq. (11) guarantees the correct quantization of the soliton as a spin $1/2$ object. Here we neglect the breaking of Lorentz symmetries, irrelevant to our discussion. The Euler-Lagrangian equations of motion for the classical, time independent, chiral field $U_0(\mathbf{r})$ are highly non-linear partial differential equations. To simplify these equations Skyrme adopted the hedgehog *ansatz* which, suitably generalized for the three flavor case, reads [40]:

$$U_0(\mathbf{r}) = \begin{pmatrix} e^{i\vec{\tau}\cdot\hat{r}F(r)} & 0 \\ 0 & 1 \end{pmatrix} \ ,$$
(23)

where $\vec{\tau}$ represents the Pauli matrices and the radial function $F(r)$ is called the chiral angle. The *ansatz* is supplemented with the boundary conditions $F(\infty) = 0$ and $F(0) = 0$ which guarantee that the configuration posseses unit baryon number. After substituting the *ansatz* in the action one finds that the classical solitonic mass is, up to a numerical factor:

$$M_{\text{soliton}} \propto \frac{F_\pi}{e} \sim \frac{F_\pi^2}{\Delta} \ ,$$
(24)

and the isoscalar radius, $\langle r^2 \rangle_{I=0} \sim 1/(F_\pi^2\,e^2) \sim 1/\Delta^2$. Interestingly, due to the non perturbative nature of the soliton, its mass turns to be dual to the vector

meson mass. It is also clear that although the vector mesons and the solitons have dual masses, they describe two very distinct types of states. The present duality is very similar to the one argued in [29]. Indeed, after introducing the collective coordinate quantization, the soliton (due to the Wess-Zumino term) describes baryonic states of half-integer spin while the vectors are spin one mesons. Here, the dual nature of the soliton with respect to the vector meson is enhanced by the fact that, in the CFL state, $\widetilde{g} \sim \Delta/F_\pi$ is expected to be substantially reduced with respect to its value in vacuum. Once the soliton is identified with the nucleon (whose density–dependent mass is denoted with $M_N(\mu)$) we predict the following relation to be independent of the matter density:

$$\frac{M_N(\mu)\, m_v(\mu)}{(2\pi F_\pi(\mu))^2} = \frac{M_N(0)\, m_v(0)}{(2\pi F_\pi(0))^2} \sim 1.05 \; . \tag{25}$$

In this way, we can relate duality to quark-hadron continuity. We considered duality, which is already present at zero chemical potential, between the soliton and the vector mesons a fundamental property of the spectrum of QCD which should persists as we increase the quark chemical potential. Should be noted that differently than in [42] we have not subtracted the energy cost to excite a soliton from the fermi sea. Since we are already considering the Lagrangian written for the excitations near the fermi surface we would expect not to consider such a corrections. In any event this is of the order μ [42] and hence negligible with respect to M_{soliton}.

We have shown that the vector mesons in the CFL phase have masses of the order of the color superconductive gap, Δ. On the other hand the solitons have masses proportional to F_π^2/Δ and hence should play no role for the physics of the CFL phase at large chemical potential. We have noted that the product of the soliton mass and the vector meson mass is independent of the gap. This behavior reflects a form of electromagnetic duality in the sense of Montonen and Olive [29]. We have predicted that the nucleon mass times the vector meson mass scales as the square of the pion decay constant at any nonzero chemical potential. In the presence of two or more scales provided by the underlying theory the spectrum of massive states shows very different behaviors which cannot be obtained by assuming a naive dimensional analysis.

4. 2 SC General Features and Effective Lagrangian

QCD with 2 massless flavors has gauge symmetry $SU_c(3)$ and global symmetry

$$SU_L(2) \times SU_R(2) \times U_V(1) \; . \tag{26}$$

At very high quark density the ordinary Goldstone phase is no longer favored compared with a superconductive one associated to the following type of di-

quark condensates:

$$\langle L^{\dagger a}\rangle \sim \langle \epsilon^{abc}\epsilon^{ij}q^{\alpha}_{Lb,i}q_{Lc,j;\alpha}\rangle \,, \qquad \langle R^{\dagger a}\rangle \sim -\langle \epsilon^{abc}\epsilon^{ij}q_{Rb,i;\dot\alpha}q^{\dot\alpha}_{Rc,j}\rangle \,, \quad (27)$$

If parity is not broken spontaneously, we have $\langle L_a\rangle = \langle R_a\rangle = f\delta^3_a$, where we choose the condensate to be in the 3rd direction of color. The order parameters are singlets under the $SU_L(2) \times SU_R(2)$ flavor transformations while possessing baryon charge $\frac{2}{3}$. The vev leaves invariant the following symmetry group:

$$[SU_c(2)] \times SU_L(2) \times SU_R(2) \times \widetilde{U}_V(1) \,, \qquad (28)$$

where $[SU_c(2)]$ is the unbroken part of the gauge group. The $\widetilde{U}_V(1)$ generator $\widetilde{B}$ is the following linear combination of the previous $U_V(1)$ generator B and the broken diagonal generator of the $SU_c(3)$ gauge group T^8:
$\widetilde{B} = B - \dfrac{2\sqrt{3}}{3}T^8 = \mathrm{diag}(0,0,1)$. The quarks with color 1 and 2 are neutral under $\widetilde{B}$ and consequently so is the condensate.

The spectrum in the 2SC state is made of 5 massive Gluons with a mass of the order of the gap, 3 massless Gluons confined (at zero temperature) into light glueballs and gapless up and down quarks in the direction (say) 3 of color.

4.1 The 5 massive Gluons

The relevant coset space G/H [43, 36] with

$$G = SU_c(3) \times U_V(1) \,, \quad \text{and} \quad H = SU_c(2) \times \widetilde{U}_V(1) \qquad (29)$$

is parameterized by:

$$V = \exp(i\xi^i X^i) \,, \qquad (30)$$

where $\{X^i\}$ $i = 1, \cdots, 5$ belong to the coset space G/H and are taken to be $X^i = T^{i+3}$ for $i = 1, \cdots, 4$ while

$$X^5 = B + \frac{\sqrt{3}}{3}T^8 = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, 0) \,. \qquad (31)$$

T^a are the standard generators of $SU(3)$. The coordinates

$$\xi^i = \frac{\Pi^i}{f} \quad i = 1, 2, 3, 4 \,, \qquad \xi^5 = \frac{\Pi^5}{\widetilde{f}} \,,$$

via Π describe the Goldstone bosons which will be absorbed in the longitudinal components of the gluons. The vevs f and $\widetilde{f}$ are, at asymptotically high densities, proportional to μ. V transforms non linearly:

$$V(\xi) \to u_V \, g \, V(\xi) \, h^{\dagger}(\xi, g, u) \, h^{\dagger}_{\widetilde{V}}(\xi, g, u) \,, \qquad (32)$$

with

$$u_V \in U_V(1) , \qquad g \in SU_c(3) ,$$
$$h(\xi, g, u) \in SU_c(2) , \qquad h_{\widetilde{V}}(\xi, g, u) \in \widetilde{U}_V(1) . \tag{33}$$

It is convenient to define the following differential form:

$$\omega_\mu = iV^\dagger D_\mu V \quad \text{with} \quad D_\mu V = (\partial_\mu - ig_s G_\mu)V , \tag{34}$$

with $G_\mu = G_\mu^m T^m$ the gluon fields while g_s is the strong coupling constant. ω transforms according to:

$$\begin{aligned}
\omega_\mu \quad \to \quad & h(\xi, g, u)\omega_\mu h^\dagger(\xi, g, u) + i\, h(\xi, g, u)\partial_\mu h^\dagger(\xi, g, u) \\
+ \quad & i\, h_{\widetilde{V}}(\xi, g, u)\partial_\mu h^\dagger_{\widetilde{V}}(\xi, g, u) .
\end{aligned}$$

We decompose ω_μ into

$$\omega_\mu^\parallel = 2S^a \text{Tr}\left[S^a \omega_\mu\right] \quad \text{and} \quad \omega_\mu^\perp = 2X^i \text{Tr}\left[X^i \omega_\mu\right] , \tag{35}$$

S^a are the unbroken generators of H, while $S^{1,2,3} = T^{1,2,3}$ and $S^4 = \widetilde{B}/\sqrt{2}$.

The most generic two derivative kinetic Lagrangian for the goldstone bosons is:

$$L = f^2 a_1 \text{Tr}\left[\omega_\mu^\perp \omega^{\mu\perp}\right] + f^2 a_2 \text{Tr}\left[\omega_\mu^\perp\right] \text{Tr}\left[\omega^{\mu\perp}\right] . \tag{36}$$
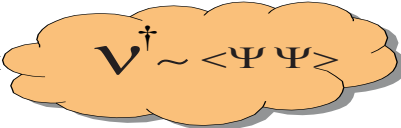
The double trace term is due to the absence of the condition for the vanishing of the trace for the broken generator X^5. It emerges naturally in the non linear realization framework at the same order in derivative expansion with respect to the single trace term. In the unitary gauge these two terms correspond to the five gluon masses [43].

4.2   The fermionic sector

For the fermions it is convenient to define the dressed fermion fields

$$\widetilde{\psi} = V^\dagger \psi , \tag{37}$$

transforming as $\widetilde{\psi} \to h_{\widetilde{V}}(\xi, g, u)h(\xi, g, u)\,\widetilde{\psi}$. ψ has the ordinary quark transformations (i.e. is a Dirac spinor). Pictorially $\widetilde{\psi}$ can be viewed as a constituent

type field or alternatively as the bare quark field ψ immersed in the diquark

cloud represented by V. The non linearly realized effective Lagrangian describing in medium fermions, gluons and their self interactions, up to two derivatives is:

$$
\begin{aligned}
L &= f^2 a_1 \text{Tr} \left[\omega_\mu^\perp \omega^{\mu\perp} \right] + f^2 a_2 \text{Tr} \left[\omega_\mu^\perp \right] \text{Tr} \left[\omega^{\mu\perp} \right] \\
&+ b_1 \overline{\widetilde{\psi}} i \gamma^\mu (\partial_\mu - i\omega_\mu^\parallel) \widetilde{\psi} + b_2 \overline{\widetilde{\psi}} \gamma^\mu \omega_\mu^\perp \widetilde{\psi} \\
&+ m_M \overline{\widetilde{\psi}^C}_i \gamma^5 (iT^2) \widetilde{\psi}_j \varepsilon^{ij} + \text{h.c.} ,
\end{aligned}
\tag{38}
$$

where $\widetilde{\psi}^C = i\gamma^2 \widetilde{\psi}^*$, $i,j = 1,2$ are flavor indices and

$$
T^2 = S^2 = \frac{1}{2} \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} .
\tag{39}
$$

Here a_1, a_2, b_1 and b_2 are real coefficients while m_M is complex. From the last two terms, representing a Majorana mass term for the quarks, we see that the massless degrees of freedom are the $\psi_{a=3,i}$. The latter possesses the correct quantum numbers to match the 't Hooft anomaly conditions [32].

4.3 The $SU_c(2)$ Glueball Lagrangian

The $SU_c(2)$ gauge symmetry does not break spontaneously and confines. Calling H a mass dimension four composite field describing the scalar glueball we can construct the following lagrangian [44]:

$$
\begin{aligned}
S_{G-ball} &= \int d^4x \left\{ \frac{c}{2} \sqrt{b} \, H^{-\frac{3}{2}} \left[\partial^0 H \partial^0 H - v^2 \partial^i H \partial^i H \right] \right. \\
&\left. - \frac{b}{2} H \log \left[\frac{H}{\hat{\Lambda}^4} \right] \right\} .
\end{aligned}
\tag{40}
$$

This Lagrangian correctly encodes the underlying $SU_c(2)$ trace anomaly. The glueballs move with the same velocity v as the underlying gluons in the 2SC color superconductor. $\hat{\Lambda}$ is related to the intrinsic scale associated with the $SU_c(2)$ theory and can be less than or of the order of few MeVs [46] [1] Once created, the light $SU_c(2)$ glueballs are stable against strong interactions but not with respect to electromagnetic processes [44]. Indeed, the glueballs couple to two photons via virtual quark loops.

$$
\Gamma \left[h \to \gamma\gamma \right] \approx 1.2 \times 10^{-2} \left[\frac{M_h}{1 \text{ MeV}} \right]^5 \text{eV} ,
\tag{41}
$$

---

[1] According to the present normalization of the glueball field $\hat{\Lambda}^4$ is $v\Lambda^4$ with Λ the intrinsic scale of $SU_c(2)$ after the coordinates have been appropriately rescaled [46, 44] to eliminate the v dependence from the action.

where $\alpha = e^2/4\pi \simeq 1/137$. For illustration purposes we consider a glueball mass of the order of 1 MeV which leads to a decay time $\tau \sim 5.5 \times 10^{-14}s$. This completes the effective Lagrangian for the 2SC state which corresponds to the Wigner-Weyl phase.

Using this Lagrangian one can estimate the $SU_c(2)$ glueball melting temperature to be [47]:

$$T_c \leq \sqrt[4]{\frac{90v^3}{2\,e\,\pi^2}}\,\hat{\Lambda} < T_{CSC} \;. \tag{42}$$

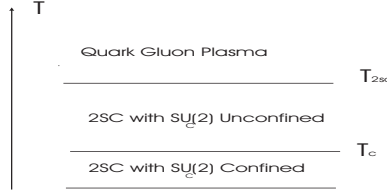Where T_{CSC} is the color superconductive transition temperature. The de-

Figure 1. A zoom of the 2SC phases as function of temperature for fixed quark chemical potential.

confining/confining $SU_c(2)$ phase transition within the color superconductive phase is second order.

5. Non Perturbative Exact Results: Anomaly Matching Conditions

The superconductive phase for $N_f = 2$ possesses the same global symmetry group as the confined Wigner-Weyl phase. The ungapped fermions have the correct global charges to match the t' Hooft anomaly conditions as shown in [32]. Specifically the $SU(2)_{L/R}{}^2 \times U(1)_V$ global anomaly is correctly reproduced in this phase due to the presence of the ungapped fermions. This is so since a quark in the 2SC case is surrounded by a diquark medium (i.e. $q\,\langle qq\rangle$) and behaves as a baryon.

$$\begin{pmatrix} u \\ d \end{pmatrix}_{color=3} \quad \sim \quad \begin{pmatrix} p \\ n \end{pmatrix} \;.$$

The validity of the t'Hooft anomaly conditions at high matter density have been investigated in [32, 33]. A delicate part of the proof presented in [33] is linked necessarily to the infrared behavior of the anomalous three point function. In particular one has to show the emergence of a singularity (i.e. a pole structure). This pole is then interpreted as due to a goldstone boson when chiral symmetry is spontaneously broken.

One might be worried that, since the chemical potential explicitly breaks Lorentz invariance, the gapless (goldstone) pole may disappear modifying the infrared structure of the three point function. This is not possible. Thanks to the Nielsen and Chadha theorem [48], not used in [33], we know that gapless excitations are always present when some symmetries break spontaneously even in the absence of Lorentz invariance[2]. Since the quark chemical potential is associated with the barionic generator which commutes with all of the non abelian global generators the number of goldstone bosons must be larger or equal to the number of broken generators. Besides all of the goldstones must have linear dispersion relations (i.e. are type I [48]). This fact not only guarantees the presence of gapless excitations (justifying the analysis made in [33] on the infrared behavior of the form factors) but demonstrates that the pole structure due to the gapless excitations needed to saturate the triangle anomaly is identical to the zero quark chemical potential one in the infrared.

It is also interesting to note that the explicit dependence on the quark chemical potential is communicated to the goldstone excitations via the coefficients of the effective Lagrangian (see [31] for a review). For example F_π is proportional to μ in the high chemical potential limit and the low energy effective theory is a good expansion in the number of derivatives which allows to consistently incorporate in the theory the Wess-Zumino-Witten term [32] and its corrections.

The validity of the anomaly matching conditions have far reaching consequences. Indeed, in the three flavor case, the conditions require the goldstone phase to be present in the hadronic as well as in the color superconductive phase supporting the quark-hadron continuity scenario [30]. At very high quark chemical potential the effective field theory of low energy modes (not to be confused with the goldstone excitations) has positive Euclidean path integral measure [49]. In this limit the CFL is also shown to be the preferred phase with the aid of the anomaly conditions. Since the fermionic theory has positive measure only at asymptotically high densities one cannot use this fact to show that the CFL is the preferred phase for moderate chemical potentials. This is possible using the anomaly constraints.

While the anomaly matching conditions are still in force at nonzero quark chemical potential [32] the *persistent mass* condition [50] ceases to be valid. Indeed a phase transition, as function of the strange quark mass, between the CFL and the 2SC phases occurs.

We recall that we can saturate the t'Hooft anomaly conditions either with massless fermionic degrees of freedom or with gapless bosonic excitations. However in absence of Lorentz covariance the bosonic excitations are not re-

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[2]Under specific assumptions which are met when Lorentz invariance is broken via the chemical potential.

stricted to be fluctuations related to scalar condensates but may be associated, for example, to vector condensates [51].

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